Statistics of Indistinguishable Particles[†]

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The wave function of a system containing identical particles takes into account the relationship between a particle's intrinsic spin and its statistical property. Specifically, the exchange of two identical particles having odd-half-integer spin results in the wave function changing sign, whereas the exchange of two identical particles having integer spin is accompanied by no such sign change. This is embodied in a term $(-1)^{2s}$, which has the value $+1$ for integer s (bosons), and -1 for odd-half-integer s (fermions), where s is the particle spin. All of this is well-known. In the nonrelativistic limit, a detailed consideration of the exchange of two identical particles shows that exchange is accompanied by a 2π reorientation that yields the $(-1)^{2s}$ term. The same bookkeeping is applicable to the relativistic case described by the proper orthochronous Lorentz group, because any proper orthochronous Lorentz transformation can be expressed as the product of spatial rotations and a boost along the direction of motion.

I. Introduction

The relationship between a particle's intrinsic spin and its statistical property lies at the heart of physical science. It is integral to the fundamental laws of nature and therefore to the numerous fields that are derivative of these laws. For example, nearly every college student has encountered the Pauli exclusion principle, which applies to identical particles that obey Fermi−Dirac statistics. This principle enables the periodic table to be rationalized by using a mnemonic in which electrons are allotted one each to spin−orbitals.

In a quantum mechanical many-body system composed of identical particles, the interchange of any two particles is accompanied by the overall wave function Ψ either changing sign or not. This can be reconciled with the help of a permutation operator P_{ij} whose role is to exchange particles i and j. Applying P_{ij} twice in succession must yield unity: $P_{ij}^2\,\Psi = \Psi$, so the eigenvalue of P_{ij}^2 is 1. It follows that $P_{ij}\Psi = \pm\Psi$, and consequently Ψ must be either symmetric or antisymmetric with respect to the exchange of identical particles. Particle spin dictates whether Ψ is symmetric (bosons) or antisymmetric (fermions). Integer s is associated with bosons, and odd-half-integer s is associated with fermions, where s is the particle spin. This relationship between a particle's spin and its statistical property can be taken for granted or proven through arduous means.

Now consider the amplitude[1]

$$\psi(x_1,x_2,...,x_i,x_j,...) \tag{1}$$

The regions between commas are referred to as slots. Each slot is associated with a particle. Convention is that particles are numbered in ascending order from left to right; i.e., the first slot corresponds to particle 1, the second to particle 2, and so on. Particle numbering and subscript numbers are not to be confused. Entries x_1, x_2, etc. denote *descriptors* (i.e., spatial coordinates and spin labels), but not particle *identity*. The latter is given by slot location, which is unaffected by permuting the contents of slots. For example, $\psi(x_2,x_3,x_1,...)$ is the amplitude for finding particle 1 described by x_2, particle 2 described by x_3, particle 3 described by x_1, and so on.

The overall wave function Ψ can be expressed as a linear combination of amplitudes such as the one in eq 1. This linear combination spans all permutations of particle identity over the field of descriptors. Leaving aside normalization, it can be written

$$\Psi = \psi(x_1,x_2,x_3,...) + c_{12}\psi(x_2,x_1,x_3,...) + c_{13}\psi(x_3,x_2,x_1,...) + ... \tag{2}$$

where $|c_{ij}| = 1$. Equation 2 includes (though not indicated explicitly) multiple interchanges that achieve all permutations of particle identity over the field of descriptors. These are expressed using products of coefficients, e.g., $c_{23}c_{13}\psi(x_3,x_1,x_2,...)$.

Referring to eq 2, when the contents of any two slots are interchanged in each and every amplitude (of course, using the same slot pair in each amplitude), Ψ must either change sign or not. Consequently, for a given group of identical particles, each c_{ij} must have the same value: $+1$ for bosons or -1 for fermions. Thus,

$$c_{ij} = (-1)^{2s} \tag{3}$$

for all $i \neq j$. Equations 2 and 3 give wave functions Ψ that are symmetric with respect to P_{ij} for integer s and antisymmetric with respect to P_{ij} for odd-half-integer s.

The above algorithm is beyond reproach in the sense that it works. For example, electronic structure theory enlists it in the form of Slater determinants or second quantization to construct antisymmetrized electron wave functions. Moreover, hard and fast rules for (identical) fermions and bosons follow. One fermion at most can occupy a given spin−orbital, whereas any number of bosons can occupy a given spin−orbital. Two or more noninteracting fermions having the same spin quantum

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number cannot be simultaneously present in the same place, whereas noninteracting bosons prefer this.

What is disquieting in the above algorithm, however, is the tacit assumption that the permutation operator P_{ij} exchanges particles. It actually exchanges the contents of slots in a mathematical expression. We shall see that particle exchange carried out explicitly differs in a subtle way that yields eq 3 directly.

Proofs of the relationship between a particle's intrinsic spin and its statistical properties have used relativistic quantum field theory or something close to it in difficulty,[2] in contrast to the simplicity of $(-1)^{2s}$. Thus, an elementary explanation has been sought, e.g., as championed by Feynman:[3] "The explanation is deep down in relativistic quantum mechanics. This probably means that we do not have a complete understanding of the fundamental principle involved. For the moment, you will just have to take it as one of the rules of the world." Attempts to rectify this situation have been varied. A proof by Sudarshan is said by him to be "simple, physical, and intuitive, but still not completely free from the complications of relativistic quantum field theory".[4] Though easier than other proofs, it is not *easy*. A proof by Berry and Robbins does not use relativistic quantum field theory,[5−7] but its topological arguments make it at least as challenging as the field theory proofs. Derivations germane to the nonrelativistic regime[8−14] have been criticized for reasons such as requiring that quantum mechanics acquires an additional postulate, assigning shape to a point particle, overlooking the fact that SO(3) is not a simply connected topological space, disregarding the light cone, mixing homotopy groups, and so on.[14−18]

Feynman revisited this matter in his Dirac Memorial Lecture, but without an elementary explanation.[19] As an afterthought he mentioned parlor tricks brought to his attention by Finkelstein.[20−22] For example, a belt is used to show that a 4π twist can be undone by moving one object around another while maintaining its orientation relative to the observation frame. On the other hand, Hilborn argues that such demonstrations are irrelevant:[18] "... this analogy is not an explanation. Nowhere does the spin of the object enter the discussion nor is it clear what the twist in the constraint has to do with the change in sign of the fermion's wave function." Duck and Sudarshan side with Hilborn, arguing that belts and the like have no bearing on exchanging identical fundamental particles because the particles and their spinors are points.[15]

The relativistic quantum field theory proof of the spin-statistics theorem is of course rigorous.[23] However, this does not mean that relativity is necessary to explain the relationship between spin and statistics. Because relativistic quantum field theory is correct (by definition subsuming all of nonrelativistic physics), it will yield correct results for a given phenomenon, regardless of whether the underlying physics is inherently relativistic or not. In this regard, it is noteworthy that the relationship between a particle's spin and its statistical properties is robust over a quite large nonrelativistic energy range, which supports the thesis that this relationship is not inherently relativistic. Taking this a step further, in the nonrelativistic regime there is, for all practical purposes, no new physics to be uncovered, so the key to understanding the $(-1)^{2s}$ term must be bookkeeping. Moreover, this must involve distinguishing labels and the exchange operation, as this is all that remains.

Because spin appears at the outset in the Dirac theory, one might question whether the symmetric treatment of space and time is essential to the spin-statistics relationship. It will be pointed out that such is not the case. This is not surprising, as spin and its properties are retained in the low-velocity limit of the Dirac equation, i.e., the (nonrelativistic) Pauli equation.[24,25] In section II, the bookkeeping issue that yields the $(-1)^{2s}$ term in the nonrelativistic limit is identified through careful examination of particle exchange. A 2π reorientation is revealed, facilitated greatly by visual aids. Application of the phase transformation operator:[26,27] $e^{-i\vec{s}\cdot\vec{\chi}}$ yields the $(-1)^{2s}$ term.

Section III provides corroborating information germane to spin-$^1/_2$. It is noted, using a mathematical object called the 3-ball, that the topological space of proper rotations in 3 dimensions, SO(3), is *not* simply connected, whereas SU(2), the group of 2×2 complex unitary matrices with determinant 1, *is* simply connected.[28−30] Thus, state vectors need not be single valued in 2π, whereas they must be single valued in 4π. Because SU(2) is a double cover of SO(3), its relationship with spin-$^1/_2$ is clear. The corresponding relativistic groups are the proper orthochronous Lorentz group (hereafter referred to as the proper Lorentz group), $SO^+(1,3)$, and the spin transformation group, $SL(2,\mathbb{C})$. The double cover of SU(2) onto SO(3) is present in the corresponding relativistic groups, with $SL(2,\mathbb{C})$ being a double cover of $SO^+(1,3)$. This identifies the spin-statistics relation vis-à-vis the proper Lorentz group as deriving from spatial rotation, just as in the nonrelativistic case.

Transformations of the proper Lorentz group vary smoothly and continuously about the identity. Any such transformation can be expressed as the product of spatial rotations and a boost in the direction of motion.[31] It follows that the nonrelativistic result can be imported into the relativistic theory via the SO(3) part of the proper Lorentz transformation. Finally, it is noted that Pauli spinors are rotors that move axes into desired locations, i.e., a classical concept.[32] Applying such rotors to spin-$^1/_2$ demonstrates the half angle transformation that accounts for exclusion.[33]

No attempt is made at a detailed derivation. We shall not venture beyond standard quantum mechanics, nor will spin's origin or deep meaning be discussed. Spin is taken as a *given*. The goal is an explanation that is accessible to a broad range of physical chemists.

II. Exchange

To examine the exchange of two identical particles, it is necessary to determine all parameters associated with the "before" and "after" configurations. Referring to Figure 1a, particle locations are indicated by $\vec{r}_a$ and $\vec{r}_b$ (open circles). Respective spin parameters (not indicated in the figure) are labeled s_a and s_b. These labels are collective: they refer to magnitudes, projections, and phases. It is often convenient to use kets such as $e^{i\delta_a}|s,m_a\rangle$, where δ_a is the phase, which plays a central role.

The pairs $\vec{r}_a,s_a$ and $\vec{r}_b,s_b$ are defined relative to a reference frame that we shall call the laboratory frame. The placement of this frame is not important. In Figure 1, its origin is where the dashed lines intersect and its z-axis can be taken as the spin quantization axis. This choice is arbitrary; it cannot affect the result. Keep in mind that $\vec{r}_a,s_a$ and $\vec{r}_b,s_b$ are not each associated with a specific particle. Each is associated with both particles because of indistinguishability. All that can be known is that one of the particles is at $\vec{r}_a$ with spin parameters s_a, whereas the other is at $\vec{r}_b$ with spin parameters s_b. It is not possible to know which particle is at a given location.

Particles 1 and 2 are now assigned to $\vec{r}_a,s_a$ and $\vec{r}_b,s_b$, respectively, as indicated in Figure 1b. Particle 1 is at $\vec{r}_a$ in spin state $e^{i\delta_a}|s,m_a\rangle$, whereas particle 2 is at $\vec{r}_b$ in spin state $e^{i\delta_b}|s,m_b\rangle$. It is convenient to set the phases δ_a and δ_b each equal to zero

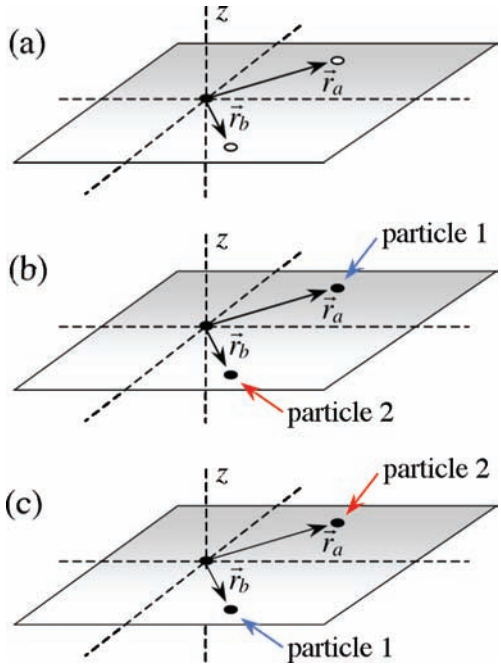

Figure 1. (a) Locations are denoted $\vec{r}_a$ and $\vec{r}_b$ (open circles), with s_a and s_b (not indicated) being the respective spin labels. The z-axis is taken as the spin quantization axis. The parameter sets: $\vec{r}_a$,s_a and $\vec{r}_b$,s_b are not each for a specific particle because the particles are indistinguishable. (b) This indicates the situation *before* exchange. Particles 1 and 2 (filled black circles labeled particle 1 and particle 2) are assigned to $\vec{r}_a$,s_a and $\vec{r}_b$,s_b, respectively. (c) This indicates the situation *after* exchange. This diagram is the same as (b) except that particles 1 and 2 have traded locations.

in this *before* configuration. One can now envision exchange as the replacement of Figure 1b by Figure 1c. Particle 1 appears at $\vec{r}_b$ in spin state $e^{i\delta_b}|s,m_b\rangle$, whereas particle 2 appears at $\vec{r}_a$ in spin state $e^{i\delta_a}|s,m_a\rangle$. However, and this point is critical, the phase difference, $\delta_a - \delta_b$, which has been set to zero *before* exchange, is not necessarily zero *following* exchange. Note that permuting the contents of slots does not take into account the possible change of $\delta_a - \delta_b$. This is the reason $(-1)^{2s}$ had to be imported ad hoc.

The scenario in Figure 1, despite its intuitive appeal, pays insufficient attention to angles. If exchange is accompanied by a 2π displacement, this will go unnoticed in the absence of an additional diagnostic, because we do not readily distinguish angles that differ by integer multiples of 2π. For example, suppose you have a model of a Cartesian frame in your office. You show it to a friend who leaves the room for a few minutes and then returns. He or she would not be able to tell whether the frame had been rotated by an integer multiple of 2π in their absence. Yet, for odd-half-integer spin and a 2π displacement, the phase difference, $\delta_a - \delta_b$, would be affected profoundly, yielding the sign change described below.

To address this issue, reference frames are assigned to each particle, as indicated in Figure 2. Particle 1 is located relative to its frame, and its frame is located relative to the laboratory, likewise for particle 2. The particles are placed on "y-axes" for viewing convenience. This incurs no loss of generality, as the placement of these additional frames is at our discretion. These frames might appear to be redundant in the sense that they do no harm but also add no information. The reason for introducing them is that they will (eventually) prove instrumental in revealing a 2π displacement.

Next, $\vec{r}_a$,s_a and $\vec{r}_b$,s_b are held fixed while the particle/frame combinations are exchanged. In effect, the particle/frame combinations enable 3D shapes to be transformed. A point object like an electron has no shape or size. It cannot rotate about its center-of-mass, so its orientation only has meaning with respect to a reference frame.

The theory of quantum mechanics is based on postulates that require the specification of coordinates. Spatial wave functions are not affected by adding integer multiples of 2π to an angular coordinate. The same is not true for odd-half-integer spin, for example, spin-$^1/_2$. Though its spinor has no spatial wave function, rotational transformation changes its phase. Specifically, the spinor is single valued in 4π and changes sign in 2π. When dealing with the exchange of identical particles, phases matter a great deal. Thus, it is necessary to distinguish integer multiples of 2π if we are to eliminate the sign ambiguity. This is the motivation behind frames 1 and 2.

In examining Figure 2, one is tempted to conclude that nothing new has been revealed. The scenario in Figure 1 has been repeated, but with frames included. Figure 2, like Figure 1, pays insufficient attention to angles—a magic wand has converted (a) into (b).

To see what is going on, the frames are connected to one another with a tether, as shown in Figure 3. I was not able to make a good drawing of the tether and its twists, so photographs will have to suffice. The role of the tether is strictly diagnostic, i.e., to answer the question: Does exchange induce a 2π reorientation? Quantum mechanics has nothing to do per se with the frames and tether. They are mere visual aids that enable us to identify a frame reorientation should it arise.

On a related matter, an intuitive way to introduce geometric phase uses a local reference frame to record the accumulation of geometric phase along a path.[34] This arises in the electronic structure of polyatomic molecules, where conical intersections are common and the associated geometric phases play a significant role.[35] The use of local reference frames in this context inspired the frames introduced in Figures 2 and 3.

A simple example of geometric phase involves vector transport on a curved surface, as with the Foucault pendulum. To see how this works, place an arrow tangent to a sphere at its north pole, and carry it along a longitude (a geodesic) to the equator without twisting it relative to the longitude. Then carry

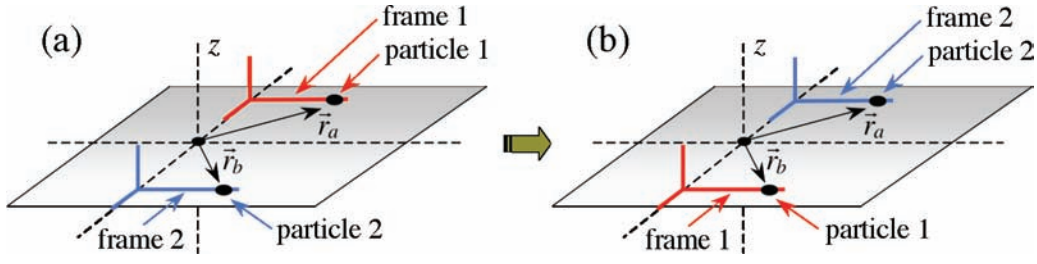

Figure 2. Particles 1 and 2 are indicated with black circles. (a) This is the situation before exchange. (b) After exchange the blue and red frames, together with particles 1 and 2, have traded locations.

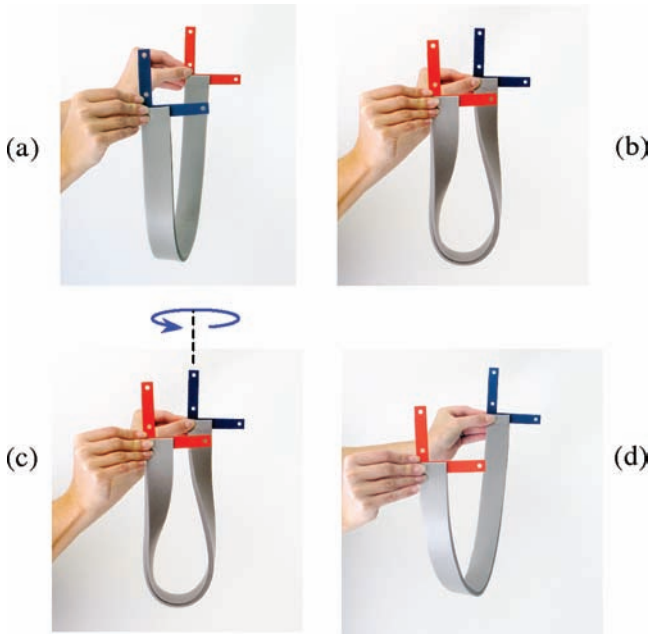

Figure 3. Blue and red frames are joined to one another with a tether. (a) Prior to exchange the blue frame is in front of the red frame. (b) After exchange the blue frame is behind the red frame and the tether has acquired a twist. (c) The twist can be undone by a 2π rotation, which gives (d).

the arrow along the equator through an azimuthal angle χ, again without twisting it. Finally, return the arrow to the north pole on a longitude without twisting it. The arrow has not been twisted on the three geodesics that comprise its path, but when it arrives back at the north pole it is displaced by the angle χ from its original orientation. The angular changes that yield χ occur at the apexes where the path switches from one geodesic to another.[36] Had the trip around the equator been a complete revolution ($\chi = 2\pi$), the arrow would have arrived back at the north pole appearing to have the same orientation as before it started its journey. In fact, it would have been reoriented by 2π relative to its original orientation.

In this case it is straightforward to show that the amount of geometric phase depends only on the enclosed area of a closed path on a (unit) sphere.[36] In molecular electronic structure, the most elementary geometric phases involve two diabats and two adiabats. As only two states are involved, these systems have spinor representations, so an enclosed area of 2π on a unit sphere (in a space where the nuclear coordinates are used as parameters) corresponds to a change of sign of the electronic wave function. In the same spirit, the use of local frames in the present paper is to reveal a 2π reorientation. Here, there is no spatial region that is avoided when taking a closed circuit, as with conical intersections[35] and the Aharonov−Bohm effect.[37−39] Nonetheless, the local frame serves the purpose of revealing angular displacement.

Figure 3 shows that the tether acquires a 2π twist when the particle/frame combinations are exchanged without rotating either relative to the laboratory. Following exchange, particle 1 can be assigned to $\vec{r}_b$,s_b with no angular displacement of frame 1 relative to the laboratory. Particle 2 is then assigned to $\vec{r}_a$,s_a, but with a 2π angular displacement of its frame relative to the laboratory. (Generality is not compromised. It would be equally valid to assign frame 2 to the laboratory. Frame 1 would then acquire a -2π displacement and the same result would be obtained.) The 2π displacement changes the phase of the ket $|s,m_a\rangle$ that has been assigned to particle 2. When $e^{-i\vec{s}\cdot\vec{\chi}}$ ($= e^{-i2\pi s_z}$) operates on $|s,m_a\rangle$ it yields the term $(-1)^{2s}$:

$$e^{-i2\pi s_z}|s,m_a\rangle = e^{-i2\pi m_a}|s,m_a\rangle = (-1)^{2m_a}|s,m_a\rangle = (-1)^{2s}|s,m_a\rangle \quad (4)$$

For this result to be general it must be independent of which axis pair is coupled by the tether (i.e., x_1/x_2, y_1/y_2, or z_1/z_2), as well as the orientation of the tether. To ensure that this is the case, two frames were constructed and Velcro tabs were attached to each axis, as well as to the ends of a plastic tether. Variations of the scenario presented in Figure 3 were tested (this time using 3 axes rather than the 2 in Figure 3). If the tether hangs vertically, as in Figure 3, the axis pair to which it is appended does not matter. Following exchange, a 2π rotation in the horizontal plane recovers the original orientation. It is understood that "original orientation" is modulo 4π, as discussed below. In other words, a 4π twist is the same as the identity. If the tether is now connected such that its width is vertical, it is found that, following exchange, a 2π rotation in the horizontal plane again recovers the original orientation. As before, this result is independent of the axis pair to which the tether is appended.

The above tests indicate that the tether width can be oriented in any of the three orthogonal directions and the original frame orientation is recovered by a single 2π rotation as in Figure 3c. The result is also independent of where the 2π rotation is taken. If the above exercise is repeated except the 2π rotation is about the x- or y-axis, the original orientation is still recovered. Thus, particle exchange includes a concomitant 2π reorientation. It does not matter where the 2π rotation is taken, so the result is independent of the choice of spin quantization axis, as required. As a final demonstration, 3 tethers were used to couple each axis pair simultaneously. The system appears quite tangled following exchange. Nonetheless, a 2π rotation about any axis undoes the twists in the tether, modulo 4π. After all is said and done, nothing unusual has been unearthed. It is just a matter of taking orientation into account with the exchange operation.

Because odd-half-integer spin is single valued in 4π, it does not matter if integer multiples of 4π are added. Referring to Figure 3b, were the blue frame to complete a full circuit around the red frame the result would be unity because exchange must be its own inverse. The 4π twist that would be incurred is consistent with this. It is pointed out in the next section (and in the Appendix) that a 2π twist cannot be removed by varying a parameter because the topological space SO(3) is not simply connected. However, a 4π twist *can* be removed by varying a parameter, so it is equivalent to the identity. Referring to Figure 3, note that it is impossible to exchange the positions of the red and blue frames (while maintaining no rotation relative to the laboratory) and not have the tether become twisted by an odd-integer multiple of 2π, which (modulo 4π) is equivalent to 2π.

This completes the main result. Section III deals with the important case of spin-$^1/_2$. Section IV is a summary.

III. Details

Section II provided an explanation of the principle whereby wave functions are either symmetric or antisymmetric with respect to the exchange of two identical particles. Arguments were based on bookkeeping and nonrelativistic physics. Application of the operator $e^{-i\vec{s}\cdot\vec{\chi}}$ to the 2π reorientation that accompanies exchange yielded the term $(-1)^{2s}$.

The Dirac equation describes an electron and its antiparticle, including the creation and annihilation of particles (field theory). It is valid throughout the regime of special relativity, whereas its low-velocity limit is the Pauli equation:[24,25]

$$i\hbar\partial_t|\psi\rangle = \left(\frac{1}{2m}(\vec{p} - e\vec{A}/c)^2 - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} + e\Phi\right)|\psi\rangle\sigma\rightarrow(5)$$

The Pauli equation describes an electron in the regime of nonrelativistic physics, including the value $g = 2$ for the reduced gyromagnetic ratio. Referring to eq 5, the σ_i are Pauli matrices, $\vec{A}$ is the magnetic vector potential, $\vec{B} = \vec{\nabla} \times \vec{A}$ is the magnetic field, Φ is the scalar electric potential, and $|\psi\rangle$ is a 2-spinor, hereafter referred to simply as a spinor. One might question whether a vestige of the relativistic theory is retained in the low-velocity limit given by eq 5, thereby accounting for the $(-1)^{2s}$ term that yields a sign change upon 2π angular displacement. I would not think so, but the question has been posed. It is also fair to ask if the explanation presented in the previous section remains valid in the relativistic regime, and if so, how might it be incorporated into the theory?

These issues can be resolved through consideration of the proper (orthochronous) Lorentz group, $SO^+(1,3)$. Its transformations preserve the Lorentz norm and are continuous about the identity, as they do not include time reversal and parity. Every proper Lorentz transformation can be expressed as the product of spatial rotations and a spacetime boost along the direction of motion.[31] If the chosen reference frame is such that motion takes place along an inconvenient direction, one simply rotates the desired axis onto the direction of motion, applies the boost, and then rotates to the final orientation. The boost is topologically trivial. The rotational part of the transformation is the standard SO(3) one encountered in nonrelativistic physics.

A matrix product describing a proper Lorentz transformation in terms of a boost and a rotation is given by[40]

$$L_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathrm{SO(3)} & \\ 0 & & & \end{bmatrix} \begin{bmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

where $\sinh\varphi = -(v/c)[1 - (v/c)^2]^{-1/2}$. The SO(3) matrix is usually expressed in terms of Euler angles. The L_p in eq 6 acts single-sidedly on the contravariant 4-vector:

$$x = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \quad (7)$$

where $x^0 = ct$, and Minkowski spacetime is assigned the Lorentzian metric (also called the Bjorken−Drell metric) with signature $(+, -, -, -)$. Applying L_p to x yields new spacetime coordinates while preserving the Lorentz norm: $\|x\|^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$. Equation 6 shows that spatial rotation can be dealt with separately from the boost. This is relevant because Section II dealt with rotational transformation.

Topology. The orientation of a 3D object is described uniquely and continuously by the group SO(3). As a topological space, SO(3) is connected, but not simply connected. Thus, state vectors need not recover their original phase when rotated by 2π, e.g., as in the case with odd-half-integer spin. One way to illustrate this "non-simply-connected" nature enlists what mathematicians call the 3-ball.[41,42] This is a solid sphere of radius π. Points within this sphere and on its surface account for all possible orientations of an asymmetric object. A few comments are given here; the Appendix provides details.

Briefly, "loops" in the 3-ball that are created by a 2π revolution of a physical object cannot be contracted to a point, meaning that SO(3) is not a simply connected topological space. On the other hand, 4π revolutions create loops that can always be contracted to a point, so this space *is* simply connected. The important thing is this: A loop that cannot be contracted to a point cannot be identified with the identity, so a state vector need not be single valued upon completing a 2π circuit. This is the origin of the geometric phase effect that arises when an electronic wave function completes a circuit around a conical intersection. Likewise, it is responsible for the Aharonov−Bohm effect.

Because there are two classes of loops, the universal covering is double. Thus, SU(2) is used, as it is a double cover of SO(3). The fact that state vectors are single valued in 4π, whereas they can change sign in 2π, is illustrated by the transformation of a Hermitian matrix that represents a 3D vector ($\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$) in the 2D complex space of SU(2):

$$\begin{bmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{bmatrix} = \begin{bmatrix} e^{-i\phi/2}\cos\theta/2 & e^{i\varphi/2}\sin\theta/2 \\ -e^{-i\phi/2}\sin\theta/2 & e^{i\phi/2}\cos\theta/2 \end{bmatrix} \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix} \begin{bmatrix} e^{i\phi/2}\cos\theta/2 & -e^{i\phi/2}\sin\theta/2 \\ e^{-i\phi/2}\sin\theta/2 & e^{-i\phi/2}\cos\theta/2 \end{bmatrix} \quad (8)$$

Note that the matrices representing $\vec{r}'$ and $\vec{r}$ are $\vec{\sigma}\cdot\vec{r}'$ and $\vec{\sigma}\cdot\vec{r}$, respectively, where the σ_i are Pauli matrices. Multiplying out the three matrices on the right-hand side and comparing the resulting matrix, element-by-element, to the left-hand side verifies that this transformation brings about 3D rotations, even though the transformation matrices use half angles. It is also clear that multiplying each of the two transformation matrices (i.e., to the left and right of the $\vec{r}$ matrix) by -1 has no effect because of the double-sided nature. That is, there are two SU(2) matrices, one the negative of the other, that yield the same 3D rotation of a physical vector. This can be seen from eq 8 by writing it as $r' = MrM^\dagger = (-M)r(-M^\dagger)$. At the same time, it is obvious that the above transformation matrices rotate a spinor by a half angle, so a 2π rotation reverses the spinor sign. These considerations illustrate the double cover.

Returning to relativity, the group of 2×2 complex unimodular matrices is called the spin transformation group, $SL(2,\mathbb{C})$. It is the relativistic counterpart of SU(2); in fact, SU(2) is a subgroup of $SL(2,\mathbb{C})$. Whereas the universal covering group for SO(3) is SU(2), the universal covering group for the proper Lorentz group $SO^+(1,3)$ is $SL(2,\mathbb{C})$, also with a double cover. This is discussed in the books of Penrose and Rindler,[28] Sternberg,[29] Richtmyer,[30] and Schutz.[43] Figure 4 summarizes relationships among the relevant groups.

The double cover of $SO^+(1,3)$ by $SL(2,\mathbb{C})$ arises from the SO(3) part of $SO^+(1,3)$, not the boost. This relationship involving spatial rotations is the same as the one between SO(3) and SU(2). Therefore, the result obtained in section II is applicable to the relativistic regime via the spatial rotation part of $SO^+(1,3)$, e.g., as indicated in eq 6. It is not restricted to nonrelativistic physics.

Rotors. Another matter that deserves comment is the role of the frames and tether shown in Figure 3. The frames and tether were introduced as a means of detecting angular displacement.

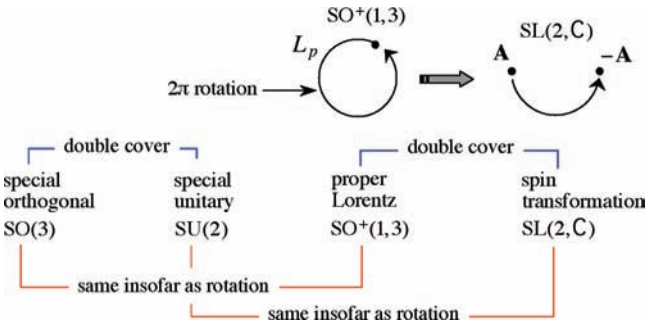

Figure 4. Spin transformations brought about with matrices **A** acting double-sidedly. The proper (orthochronous) Lorentz transformation L_p acts single-sidedly, returning the system in 2π. With SL(2,ℂ), a 2π rotation in the physical space transforms **A** to $-\mathbf{A}$.

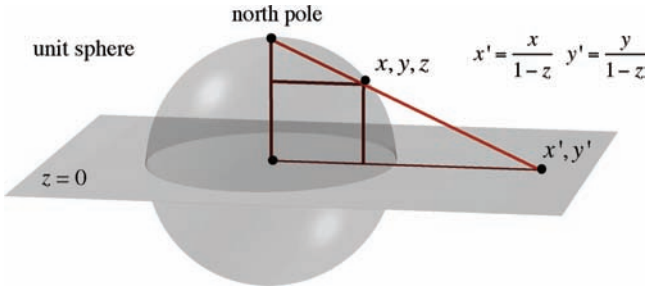

Figure 5. Unit Riemann sphere mapped onto a plane that serves as an Argand plane for the complex number $\zeta = x' + iy'$.

The 2π that was revealed applies to particles of any spin. It is particle spin that determines the sign change, or lack thereof, and consequently the statistical property. Let us consider in more detail spin-$^1/_2$, which we shall refer to hereafter simply as spin. This illustrates the transformation property used to obtain the fermion exclusion rule.

A spinor can be interpreted as an instruction to rotate a reference frame into place. A given spin and its spinor are expressed in relation to a reference frame, and it is customary to use the $+z$ direction to label "spin-up." A common expression for a spinor $|\psi\rangle$ is

$$|\psi\rangle = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{9}$$

where ξ and η are complex numbers subject to normalization: $\xi\xi^* + \eta\eta^* = 1$. Spatial wave functions are suppressed here to focus on spin's transformation properties.

This way of expressing the spinor originated with Cartan.[44,45] Referring to Figure 5, the unit sphere ($x^2 + y^2 + z^2 = 1$) is mapped onto the $z = 0$ plane using a straight line. This line has one end fixed at the north pole. The line contains a point x, y, z on the sphere and a point x', y' in the $z = 0$ plane.

The complex parameter $\zeta = x' + iy'$ keeps track of x' and y' and permits rotations and dilations in the plane to be carried out using complex algebra. The normalized version is obtained by setting $\zeta = \xi/\eta$, where ξ and η are each complex, and the pair (ξ, η) is normalized: $\xi\xi^* + \eta\eta^* = 1$. Though the spinor in eq 9 is used widely, ζ is also a spinor, albeit not a normalized one. A standard basis is the north and south poles:

$$\text{north pole:} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{south pole:} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{10}$$

Minor algebraic rearrangement (using $\zeta = x' + iy'$ and the mathematical expressions in Figure 5) yields x, y, z in terms of the parameters ξ and η and their complex conjugates:

$$\begin{aligned} x &= \sin\theta\cos\varphi = \xi\eta^* + \xi^*\eta \\ y &= \sin\theta\sin\varphi = -i(\xi\eta^* - \xi^*\eta) \\ z &= \cos\theta = \xi\xi^* - \eta\eta^* \end{aligned} \tag{11}$$

Spin components are now defined:[46]

$$s_k \equiv \frac{1}{2}\hbar n_k \tag{12}$$

where n_k is the expectation value for an arbitrary spinor:

$$n_k = \langle\psi|\hat{\sigma}_k|\psi\rangle = [\xi^* \quad \eta^*][\hat{\sigma}_k]\begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{13}$$

This yields

$$\begin{aligned} n_1 &= \xi\eta^* + \xi^*\eta \\ n_2 &= i(\xi\eta^* - \xi^*\eta) \\ n_3 &= \xi\xi^* - \eta\eta^* \end{aligned} \tag{14}$$

There is no significant difference between eqs 11 and 14. They differ only in the sign in front of the imaginary unit i, which is due to different senses of rotation, as discussed below. Thus, a correspondence is identified between a unit vector terminating on the unit sphere and spin expectation values. The spinor is normalized, so $n_1^2 + n_2^2 + n_3^2 = (\xi\xi^* + \eta\eta^*)^2 = 1$. This enables us to write the unit vector **n** as[46]

$$\mathbf{n} = \sin\theta(\cos\phi\,\boldsymbol{\sigma}_1 + \sin\phi\,\boldsymbol{\sigma}_2) + \cos\theta\,\boldsymbol{\sigma}_3 \tag{15}$$

The $\boldsymbol{\sigma}_i$'s in this expression are not operators but unit vectors. The reason for using σ to label both Pauli matrices and axes is to underscore the fact that Pauli matrices are but one representation of a vector-multiplication algebra, i.e., the Clifford algebra Cl_3.[47−51] The matrix element in eq 13 can be expressed in terms of the SU(2) transformation of the north-pole reference spinor:

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} e^{i\phi/2}\cos\theta/2 & -e^{i\phi/2}\sin\theta/2 \\ e^{-i\phi/2}\sin\theta/2 & e^{-i\phi/2}\cos\theta/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{16}$$

in which case eq 13 becomes

$$\begin{aligned} n_k = [1 \quad 0]&\begin{bmatrix} e^{-i\phi/2}\cos\theta/2 & e^{i\phi/2}\sin\theta/2 \\ -e^{-i\phi/2}\sin\theta/2 & e^{i\phi/2}\cos\theta/2 \end{bmatrix}\begin{bmatrix} \hat{\sigma}_k \end{bmatrix} \\ &\begin{bmatrix} e^{i\phi/2}\cos\theta/2 & -e^{i\varphi\phi/2}\sin\theta/2 \\ e^{-i\phi/2}\sin\theta/2 & e^{-i\phi/2}\cos\theta/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned} \tag{17}$$

This highlights the dual interpretation of the spinor, with the larger matrices acting on either the reference spinor or $\hat{\sigma}_k$. For example, suppose the spin is polarized in the $+z$ direction ($n_1 = n_2 = 0$, $n_3 = 1$ in eq 14). This can be expressed in eq 17 by using $\theta = 0$:

$$n_k = [1 \quad 0]\begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}\begin{bmatrix} \hat{\sigma}_k \end{bmatrix}\begin{bmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (18)$$

Multiplying out the matrices on the right-hand side shows that the n_k are independent of ϕ, confirming that spin components are independent of rotation in the plane perpendicular to the direction of spin polarization. Now use $\theta \neq 0$ and $\phi = 0$ in eq 17 to tilt the spin in the *xz*-plane. (Nothing would be gained by using $\phi \neq 0$.)

$$n_k = [1 \quad 0]\begin{bmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{bmatrix}\begin{bmatrix} \hat{\sigma}_k \end{bmatrix}\begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (19)$$

This yields $n_1 = \sin\theta$, $n_2 = 0$, and $n_3 = \cos\theta$, i.e., simple projections.

Equation 17 can be interpreted as keeping the σ_1, σ_2, and σ_3 axes in place and rotating the spin from the north pole onto the desired direction. This yields the projections given by eq 14. The same end is achieved by keeping the spin in place (north pole) and rotating the axes. If R acts from the left on the spinor, rotation of the σ_1, σ_2, σ_3 axes is given by $R^\dagger\sigma_k R$. Note that in this case the transformation is given by $R^\dagger\sigma_k R$ rather than the more common expression $R\sigma_k R^\dagger$. This is why eqs 11 and 14 differ by the sign of the imaginary unit i.

The use of 2D complex matrices is less transparent than the use of geometric algebra, as discussed by Lounesto,[47] Doran and Lasenby,[32] Hestenes,[48−50] and Hestenes and Sobczyk.[51] With geometric algebra, a spinor is identified as a rotor, e.g., as in classical mechanics, where it is used to locate the axes of a rigid body using Euler angles. A couple of lines of algebra show that when the plane of rotation contains the physical vector (in this case σ_3), $R^\dagger\sigma_3 R$ yields $(R^\dagger)^2\sigma_3 = \sigma_3 R^2$.[52] Thus, a half angle rotation R results in a full-angle rotation of the physical vector. The point is that one sees how a rotational transformation of the spin reference frame is imposed on the spinor. In the present case, the 2π displacement that accompanies exchange can be appended to one particle frame or the other, but not both, as it is *relative* orientation that changes. As a result, the exchange of two odd-half-integer identical particles is accompanied by a change of sign of the overall wave function. It is clear that bosons undergo no such sign change because their m_s values are integer.

The sign change of the spinor when **n** is rotated by 2π is not a purely quantum mechanical effect. It arises because the spinor alters an observable through a double-sided operation. The fact that there are only two spin eigenvalues leads to the SU(2) representation, or its geometric algebra equivalent, and the spinor sign change. It is noteworthy that the rotation of a classical object using a rotor requires that the rotor moves through $\theta/2$ as the classical object moves through θ. In other words, the same sign change is present in classical physics.[32]

IV. Summary

- The relationship between a particle's intrinsic spin and its statistical properties is embodied in the term $(-1)^{2s}$. An explanation of the spin-statistics relation in the nonrelativistic regime must be based on bookkeeping that involves particle labels and exchange. This has been achieved in a transparent manner using visual aids that reveal a 2π angular displacement that accompanies exchange. It is well-known[26,27] that a 2-spinor changes sign when subjected to a 2π spatial rotation. Thus, the $(-1)^{2s}$ term is a consequence of the 2π displacement.
- In considering spin-$^1/_2$, it is noted that the group SU(2) is a double cover of SO(3), the group of proper rotations in 3D. The topological space SO(3) is not simply connected, so wave functions need not be invariant with respect to 2π rotation, whereas SU(2) *is* simply connected. The proper Lorentz group, $SO^+(1,3)$, can be expressed as a product of rotations and a boost in the direction of motion. Its covering group, $SL(2,\mathbb{C})$, relates to $SO^+(1,3)$ in the same way that SU(2) relates to SO(3). The $(-1)^{2s}$ obtained by examining rotations in SO(3) applies to the proper Lorentz group because spatial rotations are the same in SO(3) and $SO^+(1,3)$.
- Spin is not relativistic. The low-velocity limit of the Dirac equation is the Pauli equation, which is nonrelativistic, but nonetheless contains spin. The fact that there are two eigenvalues leads to SU(2), or its geometric algebra equivalent, and exclusion.
- It is interesting to reflect on interpretation. An electron is a point particle that acquires mass by interacting with the Higgs field. It obeys an exclusion rule. This rule and the properties of spin-$^1/_2$ are one and the same. Spin-$^1/_2$ implies SU(2) or its equivalent because this accommodates 2 eigenstates for a 3D angular momentum. This leads to exclusion via the 2π discussed herein. Alternatively, the 2π, together with exclusion rules, leads to something we call spin. This perspective assigns exclusion to the particle, and spin follows. It can be said that spin and exclusion are each intrinsic. However, each implies the other, so there is no cause-effect relationship.

Appendix

Consider the orientation of an asymmetric object relative to the laboratory. A Cartesian frame is affixed to the object (with its origin at the object's center-of-mass), and one of its axes points in the laboratory $\hat{z}$ direction. This axis is now rotated through angles θ and ϕ onto a direction $\hat{k}$. The angle χ then records the amount of "twist" of the object around $\hat{k}$. These (Euler) angles define the object's orientation.

This can be parametrized using the 3-ball, which is a solid sphere of radius π, as indicated in Figure A1.[41,42] With the 3-ball, $\hat{k}$ retains its meaning as the direction defined by θ and ϕ, whereas angular displacement $\Delta\chi$ is represented by distance along a diameter.

An increase of χ is represented in the 3-ball by the length of a line superimposed on the diameter that overlaps the $\hat{k}$ direction. Let us start with $\chi = 0$, i.e., at the origin. At $\chi = \pi$, the line stretches from the origin to the surface at point *a*. Point *a* is

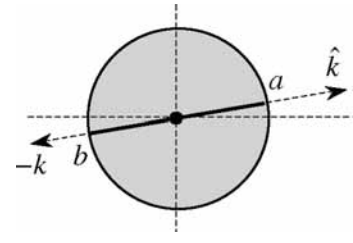

Figure A1. A cross section of the 3-ball contains the diameter associated with angle χ. When χ increases by 2π with $\hat{k}$ fixed, a straight line is produced that passes through the origin and touches the surface at the two antipodal points, *a* and *b*. The path inside the sphere can be continuously distorted and the antipodal points can be moved, but they must always remain diametrically opposed. It is impossible to continuously distort this "loop" until it is a point because of the antipodal requirement.

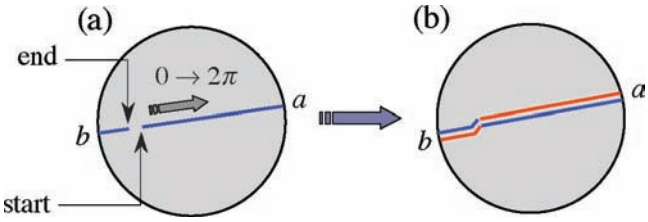

Figure A2. (a) Start and end points are identified. The gap is for viewing convenience. (b) The slight jog is for viewing convenience. For the 4π circuit, the start point is the left end of the larger blue line, whereas the end point is the right end of the shorter red line.

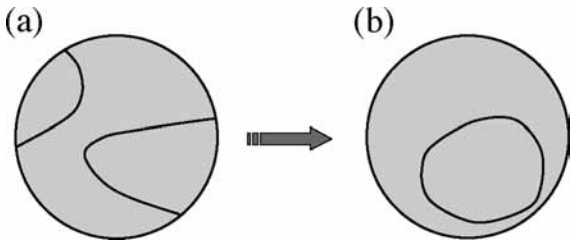

Figure A3. (a) The lines in Figure A2b are continuously distorted into these shapes. (b) Further distortion removes the left piece. This leaves the closed loop indicated in (b). This loop can be shrunk continuously to a point.

equivalent to the antipodal point b on the surface in the $-\hat{k}$ direction, because twisting an object counterclockwise in the $\hat{k}$ direction is the same as twisting it clockwise in the $-\hat{k}$ direction, and π and $-\pi$ are the same. Thus, when the system reaches a, it has also reached b.

As χ goes from π to 2π, the solid straight line in Figure A1 now goes from b to the origin. Thus, for the complete 2π circuit of χ, the line: (i) starts at the origin; (ii) goes to a; (iii) appears simultaneously at b; (iv) goes from b to the origin. It is possible to begin and end at different points along the diameter (i.e., rather than the origin), but the antipodal character must remain. It is impossible to continuously distort the path to a point because it must retain its antipodal character. This demonstrates that SO(3) is not a simply connected topological space.

Next, the object is rotated through an additional 2π. In Figure A2a the first circuit (blue) is not quite closed, to enable the start and end points to be identified. In (b) a small jog makes the second circuit (red) clear. After completion of the second circuit, the red line rejoins the blue line at the start point, yielding the closed 4π circuit.

As mentioned above, paths inside the ball can be continuously deformed as long as the antipodal nature is preserved. Referring to Figure A2b, take the red point at a and drag it counterclockwise along the surface to a point in the upper left quadrant of the figure. The red point at b must move (in synchrony) counterclockwise along the surface to preserve antipodal character. This results in Figure A3a. Now shrink the left part in Figure A3a until it vanishes. This results in the remaining curve that lies in the interior of the sphere, as indicated in Figure A3b. Clearly this loop can be shrunk to a point. This demonstrates that 4π can be continuously distorted to a point.

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